

# Vertex $F$ -algebras and their $\phi$ -coordinated modules

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## Abstract

In this paper, for every one-dimensional formal group  $F$  we formulate and study a notion of vertex  $F$ -algebra and a notion of  $\phi$ -coordinated module for a vertex  $F$ -algebra where  $\phi$  is what we call an associate of  $F$ . In the case that  $F$  is the additive formal group, vertex  $F$ -algebras are exactly ordinary vertex algebras. We give a canonical isomorphism between the category of vertex  $F$ -algebras and the category of ordinary vertex algebras. Meanwhile, for every formal group we completely determine its associates. We also study  $\phi$ -coordinated modules for a general vertex  $\mathbb{Z}$ -graded algebra  $V$  with  $\phi$  specialized to a particular associate of the additive formal group and we give a canonical connection between  $V$ -modules and  $\phi$ -coordinate modules for a vertex algebra obtained from  $V$  by Zhu's change-of-variables theorem.

## 1 Introduction

In a series of papers (see [Li3], [Li4], [Li6], [Li7], [Li9]), we have been extensively investigating various vertex algebra-like structures naturally arising from Yangians and quantum affine algebras. One of the main goals is to solve the problem (see [FJ], [EFK]), to associate “quantum vertex algebras” to quantum affine algebras. Partly motivated by Etingof-Kazhdan's theory of quantum vertex operator algebras (see [EK]), in [Li3] we developed a theory of (weak) quantum vertex algebras and we established a conceptual construction of weak quantum vertex algebras and their modules. To associate weak quantum vertex algebras to quantum affine algebras, in [Li9] we furthermore developed a theory of what we called  $\phi$ -coordinate quasi modules for weak quantum vertex algebras and established a general construction, where  $\phi$  is what we called an associate of the one-dimensional additive formal group.

In this paper, we continue to formulate and study notions of nonlocal vertex  $F$ -algebra and vertex  $F$ -algebra with  $F$  a one-dimensional formal group. It is shown that the category of vertex  $F$ -algebras is canonically isomorphic to the category of ordinary vertex algebras. We also study a notion of  $\phi$ -coordinated quasi module for a vertex  $F$ -algebra  $V$  with  $\phi$  an associate of  $F$  in the sense of [Li9], and we show that the category of  $\phi$ -coordinated quasi  $V$ -modules is canonically isomorphic to the category of  $\hat{\phi}$ -coordinated quasi  $\hat{V}$ -modules, where  $\hat{V}$  is a certain ordinary vertex algebra and  $\hat{\phi}$  is a certain associate of the additive formal group.

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In the following, we give a more detailed account of this paper. Let us start with the notion of vertex algebra, using one of several equivalent definitions (cf. [LL]). A *vertex algebra* is a vector space  $V$ , equipped with a linear map

$$Y(\cdot, x) : V \rightarrow \text{Hom}(V, V((x))) \subset (\text{End}V)[[x, x^{-1}]]$$

and equipped with a distinguished vector  $\mathbf{1} \in V$ , satisfying the following conditions:

(I) *Vacuum and creation property*

$$Y(\mathbf{1}, x)v = v, \quad Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v \quad \text{for } v \in V.$$

(II) *Weak commutativity*: For  $u, v \in V$ , there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k Y(v, x_2)Y(u, x_1). \quad (1.1)$$

(III) *Weak associativity*: For  $u, v, w \in V$ , there exists  $l \in \mathbb{N}$  such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w. \quad (1.2)$$

Note that this weak associativity relation can be viewed as an algebraic version of what physicists often call operator product expansion, while the weak commutativity relation is a version of what physicists call locality. Taking out the weak commutativity axiom from the above definition, one arrives at the notion of what was called nonlocal vertex algebra in [Li3]. (Nonlocal vertex algebras are the same as weak  $G_1$ -vertex algebras in [Li2] and are also essentially field algebras in [BK].) While vertex algebras are analogs of commutative associative algebras, nonlocal vertex algebras are analogs of noncommutative associative algebras.

One-dimensional formal groups (over  $\mathbb{C}$ ) (cf. [Ha]) are formal power series  $F(x, y) \in \mathbb{C}[[x, y]]$ , satisfying

$$F(x, 0) = x, \quad F(0, y) = y, \quad F(x, F(y, z)) = F(F(x, y), z),$$

among which the simplest is the *additive formal group*  $F_a(x, y) = x + y$ . What we called associates of a formal group  $F$  are formal series  $\phi(x, z) \in \mathbb{C}((x))[[z]]$ , satisfying

$$\phi(x, 0) = x, \quad \phi(\phi(x, z_1), z_2) = \phi(x, F(z_1, z_2)). \quad (1.3)$$

(To a certain extent, an associate of  $F$  to a formal group  $F$  is like a  $G$ -set to a group  $G$ .) Associates of  $F_a$  have been completely determined in [Li9], where two particular examples are  $\phi(x, z) = x + z = F_a(x, z)$  and  $\phi(x, z) = xe^z$ . In this present paper, we furthermore have completely determined the associates of a general one-dimensional formal group  $F$ .

Let  $F$  be a one-dimensional formal group. We define a notion of (nonlocal) vertex  $F$ -algebra by simply replacing  $x_0 + x_2 (= F_a(x_0, x_2))$  in the weak associativity relation (1.2) with  $F(x_0, x_2)$ , i.e., for  $u, v, w \in V$ , there exists  $l \in \mathbb{N}$  such that

$$F(x_0, x_2)^l Y(u, F(x_0, x_2))Y(v, x_2)w = F(x_0, x_2)^l Y(Y(u, x_0)v, x_2)w. \quad (1.4)$$

Recall that the *logarithm* of the formal group  $F$  (cf. [Ha]) is the unique series  $f(x) \in x\mathbb{C}[[x]]$  such that  $f'(0) = 1$  and  $f(F(x, y)) = f(x) + f(y)$ . Let  $(V, Y, \mathbf{1})$  be a (resp. nonlocal) vertex algebra. For  $v \in V$ , set  $Y_F(v, x) = Y(v, f(x))$ . We show that  $(V, Y_F, \mathbf{1})$  is a (resp. nonlocal) vertex  $F$ -algebra. Furthermore, we show that this gives rise to a canonical isomorphism between the category of (resp. nonlocal) vertex algebras and the category of (resp. nonlocal) vertex  $F$ -algebras.

Given a vertex  $F$ -algebra  $V$ , we define a notion of  $V$ -module in the obvious way. More generally we define a notion of  $\phi$ -coordinated  $V$ -module with  $\phi$  an associate of  $F$ , where the defining weak associativity relation reads as

$$((x_1 - x_2)^k Y(u, x_1) Y(v, x_2))|_{x_1=\phi(x_2, x_0)} = (\phi(x_2, x_0) - x_2)^k Y(Y(u, x_0)v, x_2), \quad (1.5)$$

where  $k$  is a certain nonnegative integer. Furthermore, we study a general construction of nonlocal vertex  $F$ -algebras and their  $\phi$ -coordinated modules. Let  $W$  be a general vector space and set  $\mathcal{E}(W) = \text{Hom}(W, W((x)))$ . Given an associate  $\phi$  of  $F$ , for any pair  $(a(x), b(x))$  in  $\mathcal{E}(W)$ , satisfying a certain condition, we define  $a(x)_n^\phi b(x) \in \mathcal{E}(W)$  for  $n \in \mathbb{Z}$  in terms of the generating function

$$Y_{\mathcal{E}}(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n^\phi b(x) z^{-n-1}$$

roughly by

$$“Y_{\mathcal{E}}^\phi(a(x), z)b(x) = (a(x_1)b(x))|_{x_1=\phi(x, z)}.” \quad (1.6)$$

(See Section 4 for the precise definition.) It is proved that every what we call compatible subset of  $\mathcal{E}(W)$  generates a nonlocal vertex  $F$ -algebra with  $W$  as a canonical  $\phi$ -coordinated module. This generalizes the corresponding result of [Li3].

Interestingly, the notion of  $\phi$ -coordinated module for vertex operator algebras has an intrinsic connection with Zhu's work on change-of-variables (also see the next paragraph). More specifically, let  $(V, Y, \mathbf{1}, \omega)$  be a vertex operator algebra in the sense of [FLM]. For  $v \in V$ , set  $Y[v, x] = Y(e^{xL(0)}v, e^x - 1)$ , where  $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$ . Set  $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$ , where  $c$  is the central charge of  $V$ . It was proved by Zhu ([Z1], [Z2]; cf. [Le1]) that  $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega})$  carries the structure of a vertex operator algebra. Now, let  $(W, Y_W)$  be a  $V$ -module. For  $v \in V$ , set

$$X_W(v, z) = Y_W(z^{L(0)}v, z) \in (\text{End } W)[[z, z^{-1}]].$$

We show that  $(W, X_W)$  carries the structure of a  $\phi$ -coordinated module with  $\phi(x, z) = xe^z$  for the vertex operator algebra  $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega})$ . In fact, what we have done in this paper is more general with  $V$  a nonlocal vertex  $\mathbb{Z}$ -graded algebra.

In a series of papers (see [Le2], [Le3], [Le4]), Lepowsky has extensively studied the homogenized vertex operators  $Y(x^{L(0)}v, x)$  (with  $v \in V$ ) for a general vertex operator algebra  $V$ , and obtained a very interesting new type of Jacobi identity, incorporating values of the Riemann zeta function at negative integers. The substitution  $x_1 = x_2 e^z$

and Zhu's work on change-of-variables have already entered into his study. The formal (unrigorous) relation (4.20) in [Le2] is inspirational for our introduction of the notion of  $\phi$ -coordinated module in [Li9] and it is the main motivation of Proposition 5.8 in the present paper.

In [B2], Borchers studied a notion of vertex  $G$ -algebra, which generalizes the notion of vertex algebra significantly. It was pointed out therein that the ordinary vertex algebra theory was based on the one-dimensional additive formal group and there was a vertex algebra theory associated to every formal group. Presumably, (nonlocal) vertex  $F$ -algebras belong to the family of Borchers' vertex  $G$ -algebras.

There is a very interesting paper [Lo], in which to any one-dimensional formal group, Loday associated a type of algebras, which is an operad. It was shown that for the additive formal group, the corresponding operad is that of associative algebras with derivation. It would be very interesting to relate [Lo] with the present paper.

This paper is organized as follows: In Section 2, we study associates of one-dimensional formal groups. In Section 3, we study nonlocal vertex  $F$ -algebras and vertex  $F$ -algebras. In Section 4, we study  $\phi$ -coordinated modules for nonlocal vertex  $F$ -algebras. In Section 5, we study  $\phi$ -coordinated modules for nonlocal vertex algebras with  $\phi$  specialized to  $\phi(x, z) = xe^z$ .

## 2 Associates of one-dimensional formal groups

In this section we first review the basics of one-dimensional formal groups and we then study what we called in [Li9] associates of a formal group. For every one-dimensional formal group, we completely determine its associates.

In this paper, in addition to the standard notations  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  for the integers, the rational numbers, the real numbers, and the complex numbers, respectively, we use  $\mathbb{N}$  for the nonnegative integers. All vector spaces are assumed to be over  $\mathbb{C}$  unless a different scalar field is specified otherwise.

We now follow [Ha] to recall the basics of one-dimensional formal groups.

**Definition 2.1.** Let  $R$  be a commutative and associative ring with identity. A *one-dimensional formal group* over  $R$  is a formal power series  $F(x, y) \in R[[x, y]]$ , satisfying

$$F(x, 0) = x, \quad F(0, y) = y, \quad F(x, F(y, z)) = F(F(x, y), z). \quad (2.1)$$

Throughout this paper, by a formal group we shall always mean a one-dimensional formal group. A formal group  $F(x, y)$  is said to be *commutative* if  $F(x, y) = F(y, x)$ . A fact (see [Ha]) is that formal groups over  $R$  are all commutative unless  $R$  contains a nonzero element  $a$  such that  $a^n = 0$  and  $na = 0$  for some positive integer  $n$ .

Particular examples are the *additive formal group*

$$F_a(x, y) = x + y$$

and the *multiplicative formal group*

$$F_m(x, y) = x + y + xy.$$

**Remark 2.2.** Let  $R$  be a commutative associative algebra over  $\mathbb{Q}$ . Set

$$G(R[[x]]) = \{f(x) \in R[[x]] \mid f(0) = 0, f'(0) = 1\}. \quad (2.2)$$

For  $f(x), g(x) \in G(R[[x]])$ , we have  $f(g(x)) \in G(R[[x]])$ . Furthermore,  $G(R[[x]])$  is a group with respect to the composition operation. For  $f(x) \in G(R[[x]])$ , denote the inverse of  $f(x)$  by  $f^{-1}(x)$ .

**Proposition 2.3.** Let  $R$  be a commutative associative algebra over  $\mathbb{Q}$ . Let  $f(x) \in G(R[[x]])$ . Define

$$F(x, y) = f^{-1}(f(x) + f(y)) \in R[[x, y]].$$

Then  $F(x, y)$  is a formal group over  $R$ . Furthermore, this association gives a bijection between  $G(R[[x]])$  and the set of formal groups over  $R$ .

**Definition 2.4.** Assume that  $R$  is a commutative associative algebra over  $\mathbb{Q}$ . Let  $F(x, y)$  be a formal group over  $R$ . The *logarithm of  $F$* , denoted by  $\log F$ , is defined to be the unique power series  $f(x) \in R[[x]]$  with  $f(0) = 0$  and  $f'(0) = 1$  such that

$$f(F(x, y)) = f(x) + f(y). \quad (2.3)$$

For the additive formal group and the multiplicative formal group we have

$$\log F_a = x, \quad \log F_m = \log(1 + x), \quad (2.4)$$

where by definition

$$\log(1 + x) = \sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} x^n \in \mathbb{Q}[[x]]. \quad (2.5)$$

All the basics mentioned above can be found from [Ha].

For the rest of this paper, we restrict ourselves to formal groups over  $\mathbb{C}$ . In this case, all (one-dimensional) formal groups are commutative. Recall the following notion from [Li9]:

**Definition 2.5.** Let  $F(x, y)$  be a formal group over  $\mathbb{C}$ . An *associate* of  $F(x, y)$  is a formal series  $\phi(x, z) \in \mathbb{C}((x))[[z]]$ , satisfying

$$\phi(x, 0) = x, \quad \phi(\phi(x, y), z) = \phi(x, F(y, z)). \quad (2.6)$$

To a certain extent, an associate of  $F$  to a formal group  $F$  is like a  $G$ -set to a group  $G$ . Note that every formal group  $F(x, y)$  is an associate of itself. On the other hand, it can be readily seen that  $\phi(x, z) = x$  is an associate of every formal group.

For the additive formal group  $F_a$ , we have the following explicit construction of associates due to [Li9]:

**Proposition 2.6.** For  $p(x) \in \mathbb{C}((x))$ , set

$$\phi_{p(x)}(x, z) = e^{zp(x)\frac{d}{dx}}x = \sum_{n \geq 0} \frac{z^n}{n!} \left( p(x) \frac{d}{dx} \right)^n x \in \mathbb{C}((x))[[z]].$$

Then  $\phi_{p(x)}(x, z)$  is an associate of  $F_a$ . Furthermore, every associate of  $F_a$  is of this form with  $p(x)$  uniquely determined.

Using Proposition 2.6, we obtain particular associates of  $F_a$ :  $\phi_{p(x)}(x, z) = x$  with  $p(x) = 0$ ;  $\phi_{p(x)}(x, z) = x + z$  with  $p(x) = 1$ ;  $\phi_{p(x)}(x, z) = xe^z$  with  $p(x) = x$ ;  $\phi_{p(x)}(x, z) = x(1 - zx)^{-1}$  with  $p(x) = x^2$ .

**Remark 2.7.** Let  $g(x) \in G(\mathbb{C}[[x]])$ , i.e.,  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ . We have

$$g(x)^m \in x^m \mathbb{C}[[x]] \quad \text{for } m \in \mathbb{Z}.$$

For any  $h(x) = \sum_{m \geq k} \alpha_m x^m \in \mathbb{C}((x))$  (with  $k \in \mathbb{Z}$ ,  $\alpha_m \in \mathbb{C}$ ), we have

$$h(g(x)) = \sum_{m \geq k} \alpha_m g(x)^m \in x^k \mathbb{C}[[x]] \subset \mathbb{C}((x)).$$

It is clear that the map sending  $h(x) \in \mathbb{C}((x))$  to  $h(g(x))$  is an automorphism of  $\mathbb{C}((x))$ , which preserves the subalgebra  $\mathbb{C}[[x]]$ . Furthermore, we have an automorphism of  $\mathbb{C}((x))[[z]]$ , sending  $\psi(x, z)$  to  $\psi(g(x), g(z))$  for  $\psi(x, z) \in \mathbb{C}((x))[[z]]$ . We also have an automorphism of  $\mathbb{C}((x))[[z]]$ , sending  $\psi(x, z)$  to  $\psi(x, g(z))$ .

**Proposition 2.8.** Let  $F$  be a formal group, let  $\phi$  be an associate of  $F$ , and let  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ . Set

$$\begin{aligned} F_g(x, y) &= g^{-1}(F(g(x), g(y))) \in \mathbb{C}[[x, y]], \\ \phi_g(x, z) &= g^{-1}(\phi(g(x), g(z))) \in \mathbb{C}((x))[[z]]. \end{aligned}$$

Then  $F_g$  is a formal group and  $\phi_g$  is an associate of  $F_g$ .

*Proof.* Let  $f = \log F$ . That is,  $F(x, y) = f^{-1}(f(x) + f(y))$ . Thus

$$F_g(x, y) = g^{-1}(f^{-1}(f(g(x)) + f(g(y)))).$$

In view of Proposition 2.3,  $F_g$  is a formal group with  $\log F_g = f \circ g$ .

First, note that  $\phi_g(x, z)$  is a well defined element of  $\mathbb{C}((x))[[z]]$ . Indeed, for any  $h(x) = \sum_{n \geq 0} c_n x^n \in \mathbb{C}[[x]]$ , writing  $\phi(x, z) = x + zA$  with  $A \in \mathbb{C}((x))[[z]]$ , we have

$$h(\phi(x, z)) = \sum_{n \geq 0} c_n \phi(x, z)^n = \sum_{j \geq 0} \left( \sum_{n \geq 0} \binom{n}{j} c_n x^{n-j} A^j \right) z^j \in \mathbb{C}((x))[[z]].$$

Furthermore, we have

$$\phi_g(x, 0) = g^{-1}(\phi(g(x), g(0))) = g^{-1}(\phi(g(x), 0)) = g^{-1}(g(x)) = x$$

and

$$\begin{aligned}\phi_g(\phi_g(x, y), z) &= g^{-1}(\phi(g(\phi_g(x, y)), g(z))) = g^{-1}(\phi(\phi(g(x), g(y)), g(z))) \\ &= g^{-1}(\phi(g(x), F(g(y), g(z)))) = \phi_g(x, F_g(y, z)).\end{aligned}$$

This proves that  $\phi_g(x, z)$  is an associate of  $F_g$ .  $\square$

As an immediate consequence of Proposition 2.8, we have (cf. Proposition 2.3):

**Corollary 2.9.** *Let  $F$  be a formal group over  $\mathbb{C}$  with  $\log F = f$ . For any associate  $\phi(x, z)$  of  $F_a$ , set*

$$\bar{\phi}(x, z) = f^{-1}(\phi(f(x), f(z))) \in \mathbb{C}((x))[[z]].$$

*Then  $\bar{\phi}(x, z)$  is an associate of  $F$ . Furthermore, this gives a 1-1 correspondence between the set of associates of  $F_a$  and the set of associates of  $F$ .*

The following is another connection between the associates of a general formal group and the associates of the additive formal group  $F_a$ :

**Proposition 2.10.** *Let  $F$  be a formal group over  $\mathbb{C}$  and let  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ . Then for any associate  $\phi(x, z)$  of  $F$ ,  $\phi(x, g(z))$  is an associate of  $F_g$ , where  $F_g$  is given as in Proposition 2.8. Furthermore, the map, sending  $\phi(x, z) \in \mathbb{C}((x))[[z]]$  to  $\phi(x, g(z))$ , gives a 1-1 correspondence between the set of associates of  $F$  and the set of associates of  $F_g$ .*

*Proof.* As  $\phi(x, z) \in \mathbb{C}((x))[[z]]$ ,  $g(x) \in x\mathbb{C}[[x]]$ , we see that  $\phi(x, g(z))$  exists in  $\mathbb{C}((x))[[z]]$ . We have  $\phi(x, g(0)) = \phi(x, 0) = x$  and

$$\phi(\phi(x, g(z_1)), g(z_2)) = \phi(x, F(g(z_1), g(z_2))) = \phi(x, g(F_g(z_1, z_2))).$$

Thus  $\phi(x, g(z))$  is an associate of  $F_g$ .

From the first part, we have that for any associate  $\phi(x, z)$  of  $F$ ,  $\phi(x, g(z))$  is an associate of  $F_g$  and for any associate  $\psi(x, z)$  of  $F_g$ ,  $\psi(x, g^{-1}(z))$  is an associate of  $F$ . Then the furthermore assertion follows immediately.  $\square$

**Remark 2.11.** Let  $F$  be a formal group over  $\mathbb{C}$  with  $f = \log F$  and let  $\phi$  be an associate of  $F_a$ . By Corollary 2.9 and Proposition 2.10, both  $f^{-1}(\phi(f(x), f(z)))$  and  $\phi(x, f(z))$  are associates of  $F$ . However, they are different in general. To see this, let  $F = F_m$  and let  $\phi(x, z) = F_a(x, z) = x + z$ . Note that  $f = \log F_m = \log(1 + x)$  and  $f^{-1}(x) = e^x - 1$ . We have

$$\phi(x, f(z)) = x + f(z) = x + \log(1 + z),$$

whereas

$$f^{-1}(\phi(f(x), f(z))) = f^{-1}(f(x) + f(z)) = e^{\log(1+x) + \log(1+z)} - 1 = x + z + xz.$$

Combining Propositions 2.10 and 2.6 we immediately have:

**Theorem 2.12.** *Let  $F$  be a formal group over  $\mathbb{C}$  with  $\log F = f$ . Then for any  $p(x) \in \mathbb{C}((x))$ ,  $\psi_{p(x)}(x, z) = e^{f(z)p(x)\frac{d}{dx}}x$  is an associate of  $F$ . Furthermore, every associate of  $F$  is of this form.*

The following technical result, which generalizes a result of [Li9], plays an important role in this work:

**Lemma 2.13.** *Let  $F$  be a formal group over  $\mathbb{C}$  and let  $\phi(x, z)$  be an associate of  $F$  with  $\phi(x, z) \neq x$ . Then  $q(\phi(x, z), x) \neq 0$  for any nonzero series  $q(x, y) \in \mathbb{C}((x, y))$ .*

*Proof.* Let  $f$  be the logarithm of  $F$ . By Theorem 2.12,  $\phi(x, z) = e^{f(z)p(x)\frac{d}{dx}}x$  for some nonzero  $p(x) \in \mathbb{C}((x))$ . By Proposition 2.10,  $\phi(x, f^{-1}(z))$  is an associate of  $F_a$ . As  $\phi(x, z) \neq x$ , we have  $\phi(x, f^{-1}(z)) \neq x$ . By Lemma 2.10 of [Li9], we have  $q(\phi(x, f^{-1}(z)), x) \neq 0$ . Therefore  $q(\phi(x, z), x) \neq 0$ .  $\square$

**Remark 2.14.** Recall that  $\log F_m = \log(1 + x)$ . In view of Theorem 2.12, the associates of  $F_m$  are of the form

$$\phi(x, z) = e^{(\log(1+z))p(x)\frac{d}{dx}}x$$

for  $p(x) \in \mathbb{C}((x))$ . We have particular associates of  $F_m$ :  $\phi(x, z) = x$  with  $p(x) = 0$ ,  $\phi(x, z) = x + \log(1 + z)$  with  $p(x) = 1$ ,  $\phi(x, z) = xe^{\log(1+z)} = x(1 + z)$  with  $p(x) = x$ ,  $\phi(x, z) = x(1 - x \log(1 + z))^{-1}$  with  $p(x) = x^2$ .

### 3 Nonlocal vertex $F$ -algebras and vertex $F$ -algebras

In this section we formulate and study notions of nonlocal vertex  $F$ -algebra and vertex  $F$ -algebra with  $F$  a one-dimensional formal group. When  $F = F_a$  the additive formal group, vertex  $F_a$ -algebras are simply ordinary vertex algebras, while nonlocal vertex  $F_a$ -algebras are usual nonlocal vertex algebras. As the main result of this section, we exhibit a canonical isomorphism between the category of (resp. nonlocal) vertex  $F$ -algebras and the category of ordinary (resp. nonlocal) vertex algebras. We also present some basic axiomatic properties similar to those for ordinary vertex algebras.

We begin with the notion of nonlocal vertex algebra (see [Li3]; cf. [BK], [Li2]).

**Definition 3.1.** A *nonlocal vertex algebra* is a vector space  $V$ , equipped with a linear map

$$\begin{aligned} Y(\cdot, x) : V &\rightarrow \text{Hom}(V, V((x))) \subset (\text{End}V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End}V), \end{aligned}$$



and equipped with a distinguished vector  $\mathbf{1} \in V$ , satisfying the conditions that

$$Y(\mathbf{1}, x)v = v, \quad Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v \quad \text{for } v \in V \quad (3.1)$$

(the *vacuum and creation property*), and that for  $u, v, w \in V$ , there exists  $l \in \mathbb{N}$  such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w \quad (3.2)$$

(the *weak associativity*).

Note that in (3.2), it is understood by convention that

$$\begin{aligned} Y(u, x_0 + x_2)Y(v, x_2)w &= \sum_{m, n \in \mathbb{Z}} (x_0 + x_2)^{-m-1} x_2^{-n-1} u_m v_n w \\ &= \sum_{m, n \in \mathbb{Z}, j \geq 0} \binom{-m-1}{j} x_0^{-m-1-j} x_2^{-n-1+j} u_m v_n w, \end{aligned}$$

which exists in  $V((x_0))((x_2))$ . It is important to mention that the expression

$$Y(u, x_2 + x_0)Y(v, x_2)w$$

does *not* exist as a formal series.

Now, let  $F$  be a (one-dimensional) formal group over  $\mathbb{C}$ , which is fixed throughout this section. We have  $F(x, y) = F(y, x)$ . Set  $f = \log F$ .

Recall from [Li9] (cf. [FHL]) the iota-map  $\iota_{x_1, x_2}$ , which was defined to be the unique algebra embedding of the fraction field of  $\mathbb{C}[[x_1, x_2]]$  into  $\mathbb{C}((x_1))((x_2))$  such that

$$\iota_{x_1, x_2}|_{\mathbb{C}[[x_1, x_2]]} = 1.$$

As  $F(x, y) \in \mathbb{C}[[x, y]] \subset \mathbb{C}((x))[[y]]$  with  $F(x, 0) = x$  a unit in  $\mathbb{C}((x))$ ,  $F(x, y)$ , identified as  $\iota_{x, y}F(x, y)$ , is a unit in the algebra  $\mathbb{C}((x))[[y]]$ . For the same reason,  $F(x, y)$ , identified as  $\iota_{y, x}F(x, y)$ , is a unit in the algebra  $\mathbb{C}((y))[[x]]$ . As a *convention*, for  $m \in \mathbb{Z}$  we set

$$\begin{aligned} F(x, y)^m &= (\iota_{x, y}F(x, y))^m \in \mathbb{C}((x))[[y]], \\ F(y, x)^m &= (\iota_{y, x}F(x, y))^m \in \mathbb{C}((y))[[x]]. \end{aligned}$$

Furthermore, for any  $A(x_1, x_2) \in (\text{End } W)[[x_1^{\pm 1}, x_2^{\pm 1}]]$  with  $W$  a vector space, we define  $A(F(x_0, x_2), x_2)$  and  $A(F(x_2, x_0), x_2)$  accordingly. Note that if  $A(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2)))$ ,

$$A(F(x_0, x_2), x_2) \quad \text{exists in } \text{Hom}(W, W((x_0))((x_2))).$$

If  $A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$ , in addition we have

$$A(F(x_2, x_0), x_2) \quad \text{exists in } \text{Hom}(W, W((x_2))[[x_0]]).$$

For convenience, we set

$$\begin{aligned} A(x_1, x_2)|_{x_1=F(x_0, x_2)} &= A(F(x_0, x_2), x_2), \\ A(x_1, x_2)|_{x_1=F(x_2, x_0)} &= A(F(x_2, x_0), x_2). \end{aligned}$$

For  $A(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2)))$ , we have

$$(A(x_1, x_2)|_{x_1=F(x_0, x_2)})|_{x_0=f^{-1}(f(x_1)-f(x_2))} = A(x_1, x_2), \quad (3.3)$$

where the substitution  $x_0 = f^{-1}(f(x_1) - f(x_2))$  means

$$x_0^m = \iota_{x_1, x_2} (f^{-1}(f(x_1) - f(x_2)))^m \quad \text{for all } m \in \mathbb{Z}.$$

**Definition 3.2.** We define a notion of *nonlocal vertex  $F$ -algebra*, using all the axioms in Definition 3.1 except that weak associativity is replaced by the property that for  $u, v, w \in V$ , there exists  $l \in \mathbb{N}$  such that

$$F(x_0, x_2)^l Y(u, F(x_0, x_2)) Y(v, x_2) w = F(x_0, x_2)^l Y(Y(u, x_0) v, x_2) w \quad (3.4)$$

(the *weak  $F$ -associativity*).

Furthermore, we formulate a notion of vertex  $F$ -algebra as follows:

**Definition 3.3.** A *vertex  $F$ -algebra* is a nonlocal vertex  $F$ -algebra  $V$  satisfying the *weak commutativity*: For  $u, v \in V$ , there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k Y(v, x_2) Y(u, x_1). \quad (3.5)$$

For convenience we formulate two technical lemmas.

**Lemma 3.4.** Let  $W$  be a vector space, let  $a(x), b(x) \in \text{Hom}(W, W((x)))$ , and let  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) \neq 0$ . Let  $k$  be a nonnegative integer. Then

$$(g(x_1) - g(x_2))^k a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

if and only if

$$(x_1 - x_2)^k a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

On the other hand,

$$(g(x_1) - g(x_2))^k a(x_1) b(x_2) = (g(x_1) - g(x_2))^k b(x_2) a(x_1)$$

if and only if

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1).$$

*Proof.* As  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) \neq 0$ , we have  $g(x_1) - g(x_2) = (x_1 - x_2)h(x_1, x_2)$  for some  $h(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  with  $h(0, 0) = g'(0) \neq 0$ . Then

$$(g(x_1) - g(x_2))^n = (x_1 - x_2)^n h(x_1, x_2)^n$$

for  $n \in \mathbb{N}$ . Note that as  $h(0, 0) \neq 0$ ,  $h(x_1, x_2)$  is a unit in the algebra  $\mathbb{C}[[x_1, x_2]]$ , i.e., there exists  $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  such that  $h(x_1, x_2)q(x_1, x_2) = 1$ . It then follows immediately.  $\square$

The following can be proved similarly:

**Lemma 3.5.** *Let  $V$  and  $W$  be vector spaces and let*

$$\begin{aligned} Y(\cdot, x) &: V \rightarrow \text{Hom}(V, V((x))), \\ Y_W(\cdot, x) &: V \rightarrow \text{Hom}(W, W((x))) \end{aligned}$$

*be linear maps. Let  $u, v \in V$  and let  $k \in \mathbb{N}$ . Assume*

$$(g(x_1) - g(x_2))^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(V, V((x_1, x_2)))$$

*and*

$$\begin{aligned} & (g(F(x_0, x_2)) - g(x_2))^k Y_W(Y(u, x_0)v, x_2) \\ &= [(g(x_1) - g(x_2))^k Y_W(u, x_1) Y_W(v, x_2)]|_{x_1=F(x_2, x_0)} \end{aligned}$$

*for some  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) \neq 0$ . Then the same relations hold for every such  $g(x)$ .*

The following is an explicit connection between (resp. nonlocal) vertex  $F$ -algebras and ordinary (resp. nonlocal) vertex algebras:

**Proposition 3.6.** *Let  $F$  be a formal group over  $\mathbb{C}$ , let  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ , and let  $F_g$  be the formal group defined by*

$$F_g(x, y) = g^{-1}(F(g(x), g(y)))$$

*as in Proposition 2.8. Let  $V$  be a (resp. nonlocal) vertex  $F$ -algebra. For  $v \in V$ , set*

$$Y_g(v, x) = Y(v, g(x)).$$

*Then  $(V, Y_g, \mathbf{1})$  carries the structure of a (resp. nonlocal) vertex  $F_g$ -algebra.*

*Proof.* As  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ , for  $v \in V$  we have

$$Y_g(v, x) = Y(v, g(x)) \in \text{Hom}(V, V((x))).$$

We also have  $Y_g(\mathbf{1}, x)v = Y(\mathbf{1}, g(x))v = v$ , and

$$Y_g(v, x)\mathbf{1} = Y(v, g(x))\mathbf{1} = \sum_{n \geq 0} g(x)^n v_{-n-1}\mathbf{1},$$

which implies

$$Y_g(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y_g(v, x)\mathbf{1} = v.$$

Furthermore, let  $u, v, w \in V$ . There exists  $l \in \mathbb{N}$  such that

$$F(z_0, z_2)^l Y(u, F(z_0, z_2)) Y(v, z_2) w = F(z_0, z_2)^l Y(Y(u, z_0)v, z_2) w.$$

Write  $g^{-1}(x) = xh(x)$  with  $h(x) \in \mathbb{C}[[x]]$ . Multiplying the both sides of the equation above by  $h(F(z_0, z_2))^l$  which lies in  $\mathbb{C}[[z_0, z_2]]$ , we get

$$(g^{-1}(F(z_0, z_2)))^l Y(u, F(z_0, z_2)) Y(v, z_2) w = (g^{-1}(F(z_0, z_2)))^l Y(Y(u, z_0)v, z_2) w.$$

Substituting  $z_0 = g(x_0)$  and  $z_2 = g(x_2)$  into the above equation, we obtain

$$\begin{aligned} & (g^{-1}(F(g(x_0), g(x_2))))^l Y(u, F(g(x_0), g(x_2))) Y(v, g(x_2)) w \\ &= (g^{-1}(F(g(x_0), g(x_2))))^l Y(Y(u, g(x_0))v, g(x_2)) w, \end{aligned}$$

which is

$$\begin{aligned} & F_g(x_0, x_2)^l Y(u, g(F_g(x_0, x_2))) Y(v, g(x_2)) w \\ &= F_g(x_0, x_2)^l Y(Y(u, g(x_0))v, g(x_2)) w. \end{aligned}$$

Namely,

$$F_g(x_0, x_2)^l Y_g(u, F_g(x_0, x_2)) Y_g(v, x_2) w = F_g(x_0, x_2)^l Y_g(Y_g(u, x_0)v, x_2) w.$$

This proves that  $(V, Y_g, \mathbf{1})$  carries the structure of a nonlocal vertex  $F_g$ -algebra.

Now, assume that  $V$  is a vertex  $F$ -algebra. Let  $u, v \in V$ . There exists a nonnegative integer  $k$  such that (3.5) holds. In view of Lemma 3.4 we have

$$(g^{-1}(x_1) - g^{-1}(x_2))^k Y(u, x_1) Y(v, x_2) = (g^{-1}(x_1) - g^{-1}(x_2))^k Y(v, x_2) Y(u, x_1).$$

With substitution  $x_1 = g(z_1)$ ,  $x_2 = g(z_2)$ , we get

$$(z_1 - z_2)^k Y_g(u, z_1) Y_g(v, z_2) = (z_1 - z_2)^k Y_g(v, z_2) Y_g(u, z_1).$$

Thus  $(V, Y_g, \mathbf{1})$  is a vertex  $F_g$ -algebra. □

Let  $V$  be an ordinary (resp. nonlocal) vertex algebra. For  $v \in V$ , set

$$Y_F(v, x) = Y(v, f(x)).$$

It follows from Proposition 3.6 that  $(V, Y_F, \mathbf{1})$  carries the structure of a (resp. non-local) vertex  $F$ -algebra. Denote this (resp. nonlocal) vertex  $F$ -algebra by  $V_F$ .

To summarize, in terms of categories we have:

**Theorem 3.7.** *Let  $F$  be a formal group over  $\mathbb{C}$  with  $\log F = f$ . The map  $\mathcal{F}$ , defined by  $\mathcal{F}(V) = V_F$  for every (resp. nonlocal) vertex algebra  $V$  and  $\mathcal{F}(\theta) = \theta$  for every homomorphism  $\theta$  of (resp. nonlocal) vertex algebras, is an isomorphism from the category of (resp. nonlocal) vertex algebras and the category of (resp. nonlocal) vertex  $F$ -algebras.*

*Proof.* Let  $\theta : U \rightarrow V$  be a homomorphism of nonlocal vertex algebras. We have  $\theta(\mathbf{1}) = \mathbf{1}$  and

$$\theta(Y_F(u, x)u') = \theta(Y(u, f(x))u') = Y(\theta(u), f(x))\theta(u') = Y_F(\theta(u), x)\theta(u')$$

for  $u, u' \in U$ . Thus  $\theta$  is also a homomorphism of nonlocal vertex  $F$ -algebras from  $U_F$  to  $V_F$ . It follows that  $\mathcal{F}$  is a functor from the category of (resp. nonlocal) vertex algebras to the category of (resp. nonlocal) vertex  $F$ -algebras. On the other hand, by Proposition 3.6, for each (resp. nonlocal) vertex  $F$ -algebra  $(K, Y, \mathbf{1})$ ,  $(K, Y(\cdot, f^{-1}(x)), \mathbf{1})$  is an ordinary (resp. nonlocal) vertex algebra. Then we have a functor from the category of (resp. nonlocal) vertex  $F$ -algebras to the category of (resp. nonlocal) vertex algebras, sending each nonlocal vertex  $F$ -algebra  $(K, Y, \mathbf{1})$  to  $(K, Y(\cdot, f^{-1}(x)), \mathbf{1})$ . It follows immediately that  $\mathcal{F}$  is an isomorphism.  $\square$

**Example 3.8.** We here generalize Borchers' construction of (nonlocal) vertex algebras. Suppose that  $F$  is a formal group over  $\mathbb{C}$  with  $\log F = f$ . Let  $A$  be an associative algebra (over  $\mathbb{C}$ ) with identity and let  $D$  be a derivation of  $A$ . Define

$$Y_F(a, x)b = (e^{f(x)D}a)b \quad \text{for } a, b \in A. \quad (3.6)$$

It is known (see [B1], [B2]) that for  $F = F_a$ ,  $(A, Y_{F_a}, \mathbf{1})$  carries the structure of a nonlocal vertex algebra, which is a vertex algebra if and only if  $A$  is commutative. Then by Proposition 3.6, for a general  $F$ ,  $(A, Y_F, \mathbf{1})$  carries the structure of a nonlocal vertex  $F$ -algebra. Furthermore,  $(A, Y_F, \mathbf{1})$  is a vertex  $F$ -algebra if and only if  $A$  is commutative.

Next we present another version of weak  $F$ -associativity.

**Proposition 3.9.** *In the definition of a nonlocal vertex  $F$ -algebra, weak  $F$ -associativity can be replaced by the property that for  $u, v \in V$ , there exists  $k \in \mathbb{N}$  such that*

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) \in \text{Hom}(V, V((x_1, x_2))), \quad (3.7)$$

and

$$\begin{aligned} & ((x_1 - x_2)^k Y(u, x_1)Y(v, x_2))|_{x_1=F(x_2, x_0)} \\ &= (F(x_2, x_0) - x_2)^k Y(Y(u, x_0)v, x_2). \end{aligned} \quad (3.8)$$

Note that as  $(Y(u, x_1)Y(v, x_2))|_{x_1=F(x_2, x_0)}$  does not exist as a formal series, (3.7) is a precondition for (3.8) to make sense.

*Proof.* First, assuming the very property we prove weak  $F$ -associativity. Let  $u, v, w \in V$ . Let  $k \in \mathbb{N}$  be such that (3.7) and (3.8) hold. In view of Lemma 3.4, we have

$$(f(x_1) - f(x_2))^k Y(u, x_1)Y(v, x_2) \in \text{Hom}(V, V((x_1, x_2))).$$

Furthermore, there exists another nonnegative integer  $l$  such that

$$x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w \in V[[x_1, x_2]][x_2^{-1}],$$

involving only nonnegative powers of  $x_1$ . Then

$$\begin{aligned} & (x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) |_{x_1=F(x_2, x_0)} \\ &= (x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) |_{x_1=F(x_0, x_2)} \\ &= F(x_0, x_2)^l (f(F(x_0, x_2)) - f(x_2))^k Y(u, F(x_0, x_2)) Y(v, x_2) w \\ &= F(x_0, x_2)^l f(x_0)^k Y(u, F(x_0, x_2)) Y(v, x_2) w. \end{aligned}$$

On the other hand, writing  $f(x_1) - f(x_2) = (x_1 - x_2)q(x_1, x_2)$  with  $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ , using (3.8) we have

$$\begin{aligned} & (x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) |_{x_1=F(x_2, x_0)} \\ &= (x_1^l q(x_1, x_2)^k (x_1 - x_2)^k Y(u, x_1) Y(v, x_2) w) |_{x_1=F(x_2, x_0)} \\ &= F(x_2, x_0)^l f(x_0)^k Y(Y(u, x_0) v, x_2) w. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & F(x_0, x_2)^l f(x_0)^k Y(u, F(x_0, x_2)) Y(v, x_2) w \\ &= F(x_0, x_2)^l f(x_0)^k Y(Y(u, x_0) v, x_2) w, \end{aligned} \tag{3.9}$$

recalling that  $F(x_0, x_2) = F(x_2, x_0)$ . By canceling the factor  $f(x_0)^k$  we obtain (3.4).

Now, assume weak  $F$ -associativity holds. Let  $u, v \in V$ . There exists  $k \in \mathbb{N}$  such that  $x_0^k Y(u, x_0) v \in V[[x_0]]$ , which implies  $f(x_0)^k Y(u, x_0) v \in V[[x_0]]$ . Let  $w \in V$  be arbitrarily fixed. There exists  $l \in \mathbb{N}$  such that (3.4) holds. Then we have (3.9). We see that the left-hand side of (3.9) involves only nonnegative powers of  $x_0$ , so does the right-hand side. As

$$\begin{aligned} & (x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) |_{x_1=F(x_0, x_2)} \\ &= F(x_0, x_2)^l f(x_0)^k Y(u, F(x_0, x_2)) Y(v, x_2) w, \end{aligned}$$

we have

$$(x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) |_{x_1=F(x_0, x_2)} \in V[[x_0, x_2]][x_2^{-1}].$$

By substitution  $x_0 = f^{-1}(f(x_1) - f(x_2))$  (recall (3.3)), we get

$$x_1^l(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w \in V[[x_1, x_2]][x_2^{-1}].$$

Thus

$$(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w \in V((x_1, x_2)).$$

As  $k$  depends on  $u$  and  $v$ , but not  $w$ , we have

$$(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) \in \text{Hom}(V, V((x_1, x_2))).$$

Furthermore, we have

$$\begin{aligned} & (x_1^l (f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) \big|_{x_1=F(x_0, x_2)} \\ &= (x_1^l (f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) \big|_{x_1=F(x_2, x_0)} \\ &= F(x_2, x_0)^l [(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w] \big|_{x_1=F(x_2, x_0)}. \end{aligned}$$

Combining this with (3.9) we get

$$\begin{aligned} & F(x_0, x_2)^l f(x_0)^k Y(Y(u, x_0)v, x_2) w \\ &= F(x_0, x_2)^l [(f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w] \big|_{x_1=F(x_2, x_0)}. \end{aligned}$$

Multiplying both sides by  $\iota_{x_2, x_0} F(x_0, x_2)^{-l}$ , we obtain

$$\begin{aligned} & f(x_0)^k Y(Y(u, x_0)v, x_2) w \\ &= ((f(x_1) - f(x_2))^k Y(u, x_1) Y(v, x_2) w) \big|_{x_1=F(x_2, x_0)}. \end{aligned}$$

With  $k$  independent of  $w$ , we have (3.8).  $\square$

**Remark 3.10.** Note that in the special case with  $F = F_a$ , Proposition 3.9 (for ordinary nonlocal vertex algebras) follows immediately from Lemma 2.9 of [LTW] with  $W = V$  (the adjoint module) and  $\sigma = 1$  (the identity automorphism).

The following is an analog of a result in the theory of vertex algebras (cf. [Li2]):

**Lemma 3.11.** *Let  $V$  be a nonlocal vertex  $F$ -algebra. Define a linear operator  $\mathcal{D}$  on  $V$  by*

$$\mathcal{D}v = v_{-2}\mathbf{1} = \left( \frac{d}{dx} Y(v, x) \mathbf{1} \right) \big|_{x=0} \quad \text{for } v \in V.$$

Set  $f = \log F$ . Then

$$\begin{aligned} Y(v, x) \mathbf{1} &= e^{f(x)\mathcal{D}} v, \\ [\mathcal{D}, Y(v, x)] &= Y(\mathcal{D}v, x) = \frac{1}{f'(x)} \frac{d}{dx} Y(v, x) \quad \text{for } v \in V. \end{aligned} \quad (3.10)$$

*Proof.* For  $v \in V$ , set  $\hat{Y}(v, x) = Y(v, f^{-1}(x))$ . By Proposition 3.6,  $(V, \hat{Y}, \mathbf{1})$  is an ordinary nonlocal vertex algebra. Let  $\hat{\mathcal{D}}$  be the operator on  $V$  defined by  $\hat{\mathcal{D}}v = \lim_{x \rightarrow 0} \frac{d}{dx} \hat{Y}(v, x) \mathbf{1}$  for  $v \in V$ . We have  $\hat{\mathcal{D}} = \mathcal{D}$  as

$$\hat{\mathcal{D}}(v) = \lim_{x \rightarrow 0} \frac{d}{dx} \hat{Y}(v, x) \mathbf{1} = \lim_{x \rightarrow 0} \frac{d}{dx} Y(v, f^{-1}(x)) \mathbf{1} = v_{-2} \mathbf{1} = \mathcal{D}(v).$$

Using those  $\mathcal{D}$ -properties for nonlocal vertex algebras (see [Li2]) we obtain

$$\begin{aligned} Y(v, x)\mathbf{1} &= \hat{Y}(v, f(x))\mathbf{1} = e^{f(x)\hat{\mathcal{D}}}v = e^{f(x)\mathcal{D}}v, \\ [\mathcal{D}, Y(v, x)] &= [\hat{\mathcal{D}}, \hat{Y}(v, f(x))] = \hat{Y}(\mathcal{D}v, f(x)) = Y(\mathcal{D}v, x), \end{aligned}$$

and

$$Y(\mathcal{D}v, x) = \hat{Y}(\hat{\mathcal{D}}v, f(x)) = \left( \frac{d}{dz} \hat{Y}(v, z) \right) \Big|_{z=f(x)} = \frac{1}{f'(x)} \frac{d}{dx} Y(v, x),$$

as desired.  $\square$

The following is an analog of a well known result in vertex algebra theory (see [FHL], [DL], [Li1]):

**Proposition 3.12.** *A vertex  $F$ -algebra can be defined as a vector space  $V$  equipped with a linear map  $Y(\cdot, x) : V \rightarrow \text{Hom}(V, V((x)))$ , a vector  $\mathbf{1} \in V$ , and a linear operator  $\mathcal{D}$  on  $V$ , such that (3.1), (3.10), and weak commutativity hold.*

*Proof.* We only need to prove that  $V$  is a vertex  $F$ -algebra under the very assumptions. For  $v \in V$ , set  $\hat{Y}(v, x) = Y(v, f^{-1}(x))$ , where  $f(x) = \log F$ . We have  $\hat{Y}(\mathbf{1}, x)v = Y(\mathbf{1}, f^{-1}(x))v = v$ , and

$$\hat{Y}(v, x)\mathbf{1} = Y(v, f^{-1}(x))\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} \hat{Y}(v, x)\mathbf{1} = v,$$

noticing that  $f^{-1}(x) \in x\mathbb{C}[[x]]$  with  $(f^{-1})'(0) = 1$ . Let  $u, v \in V$ . There exists  $k \in \mathbb{N}$  such that (3.5) holds. From the second half of the proof of Proposition 3.6, we have

$$(z_1 - z_2)^k \hat{Y}(u, z_1) \hat{Y}(v, z_2) = (z_1 - z_2)^k \hat{Y}(v, z_2) \hat{Y}(u, z_1).$$

Furthermore, using (3.10) we get

$$\begin{aligned} [\mathcal{D}, \hat{Y}(v, x)] &= [\mathcal{D}, Y(v, f^{-1}(x))] = \left( \frac{1}{f'(z)} \frac{d}{dz} Y(v, z) \right) \Big|_{z=f^{-1}(x)} \\ &= \frac{d}{dx} Y(v, f^{-1}(x)) = \frac{d}{dx} \hat{Y}(v, x). \end{aligned}$$

By Proposition 2.2.4 of [Li1],  $(V, \hat{Y}, \mathbf{1})$  is a vertex algebra. Now, it follows from Proposition 3.6 that  $(V, Y, \mathbf{1})$  is a vertex  $F$ -algebra.  $\square$

Next, we discuss a Jacobi identity for vertex  $F$ -algebras. Set  $f = \log F$  as before. Recall that  $f(x) \in \mathbb{C}[[x]]$  with  $f(0) = 0$  and  $f'(0) = 1$ . We have

$$\begin{aligned} f(x_0)^{-1} \delta \left( \frac{f(x_1) - f(x_2)}{f(x_0)} \right) &- f(x_0)^{-1} \delta \left( \frac{f(x_2) - f(x_1)}{-f(x_0)} \right) \\ &= f(x_1)^{-1} \delta \left( \frac{f(x_2) + f(x_0)}{f(x_1)} \right). \end{aligned} \tag{3.11}$$



To see the existence in  $\mathbb{C}[[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]]$  of the three terms, let us consider the first term. By definition, we have

$$\begin{aligned} f(x_0)^{-1} \delta \left( \frac{f(x_1) - f(x_2)}{f(x_0)} \right) &= \sum_{n \in \mathbb{Z}} f(x_0)^{-n-1} (f(x_1) - f(x_2))^n \\ &= \sum_{n \in \mathbb{Z}} \sum_{j \geq 0} \binom{n}{j} (-1)^j f(x_0)^{-n-1} f(x_1)^{n-j} f(x_2)^j, \end{aligned}$$

where

$$f(x_2)^j \in x_2^j \mathbb{C}[[x_2]], \quad f(x_1)^{n-j} \in x_1^{n-j} \mathbb{C}[[x_1]], \quad f(x_0)^{-n-1} \in x_0^{-n-1} \mathbb{C}[[x_0]].$$

We see that the expression exists, by first considering the coefficient of a fixed power of  $x_2$ , to reduce to finitely many  $j$ . Furthermore, for each fixed  $j$ , we have

$$\begin{aligned} \sum_{n \geq 0} \binom{n}{j} f(x_0)^{-n-1} f(x_1)^{n-j} &\in \mathbb{C}((x_0))((x_1)), \\ \sum_{n < 0} \binom{n}{j} f(x_0)^{-n-1} f(x_1)^{n-j} &\in \mathbb{C}((x_1))((x_0)). \end{aligned}$$

Using Proposition 3.6 we immediately get:

**Proposition 3.13.** *In the definition of a vertex  $F$ -algebra, weak  $F$ -associativity and weak commutativity can be replaced by the property that for  $u, v \in V$ ,*

$$\begin{aligned} &f(x_0)^{-1} \delta \left( \frac{f(x_1) - f(x_2)}{f(x_0)} \right) Y(u, x_1) Y(v, x_2) \\ &\quad - f(x_0)^{-1} \delta \left( \frac{f(x_2) - f(x_1)}{-f(x_0)} \right) Y(v, x_2) Y(u, x_1) \\ &= f(x_1)^{-1} \delta \left( \frac{f(x_2) + f(x_0)}{f(x_1)} \right) Y(Y(u, x_0)v, x_2) \end{aligned} \tag{3.12}$$

(the Jacobi  $F$ -identity), where  $f = \log F$ .

## 4 $\phi$ -coordinated modules for a vertex $F$ -algebra

In this section, we study  $\phi$ -coordinated (quasi) modules for a vertex  $F$ -algebra with  $\phi$  an associate of  $F$ . As the main results of this section, we give a canonical connection between  $\phi$ -coordinated (quasi) modules for nonlocal vertex algebras and for nonlocal vertex  $F$ -algebras, and we give a general construction of nonlocal vertex  $F$ -algebras and their  $\phi$ -coordinated (quasi) modules.

We begin with the notion of module for a nonlocal vertex algebra (see [Li2]).

**Definition 4.1.** Let  $V$  be a nonlocal vertex algebra. A  $V$ -module is a vector space  $W$  equipped with a linear map

$$\begin{aligned} Y_W(\cdot, x) : V &\rightarrow \text{Hom}(W, W((x))) \subset (\text{End} W)[[x, x^{-1}]] \\ v &\mapsto Y_W(v, x), \end{aligned}$$

satisfying the conditions that

$$Y_W(\mathbf{1}, x) = 1_W \text{ (the identity operator on } W)$$

and that for  $u, v \in V$ ,  $w \in W$ , there exists  $l \in \mathbb{N}$  such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_0 + x_2)^l Y_W(Y(u, x_0) v, x_2) w. \quad (4.1)$$

We define a notion of *quasi  $V$ -module* (cf. [Li3], [Li9]) by replacing the above weak associativity with the property that for  $u, v \in V$ , there exists a nonzero power series  $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  such that

$$q(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

and

$$(q(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2))|_{x_1=x_2+x_0} = q(x_0 + x_2, x_2) Y_W(Y(u, x_0) v, x_2).$$

**Remark 4.2.** From Lemma 2.9 of [LTW], in the definition of a module for a nonlocal vertex algebra  $V$ , weak associativity can be replaced by the property that for  $u, v \in V$ , there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

and

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1=x_2+x_0} = x_0^k Y_W(Y(u, x_0) v, x_2).$$

In view of this, a  $V$ -module is indeed a quasi  $V$ -module.

Now, let  $F$  be a (one-dimensional) formal group over  $\mathbb{C}$ , which is fixed for the rest of this section. Set  $f = \log F$ .

**Definition 4.3.** Let  $V$  be a nonlocal vertex  $F$ -algebra and let  $\phi$  be an associate of  $F$ . We define a notion of  *$\phi$ -coordinated quasi  $V$ -module*, using all the axioms in Definition 4.1 except that weak associativity is replaced by the property that for  $u, v \in V$ , there exists  $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  such that  $q(\phi(x_2, x_0), x_2) \neq 0$ ,

$$q(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

and

$$\begin{aligned} &(q(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)} \\ &= q(\phi(x_2, x_0), x_2) Y_W(Y(u, x_0) v, x_2) \end{aligned} \quad (4.2)$$

(the *weak  $\phi$ -associativity*). We define a notion of  *$\phi$ -coordinated  $V$ -module* by strengthening weak  $\phi$ -associativity with  $q(x_1, x_2)$  required to be a polynomial of the form  $(x_1 - x_2)^k$  with  $k \in \mathbb{N}$ . Furthermore, we call an  $F$ -coordinated  $V$ -module a  $V$ -module.

We have the following analog of Theorem 3.7:

**Proposition 4.4.** *Let  $F$  be a formal group over  $\mathbb{C}$  with  $f = \log F$ , let  $\phi$  be an associate of  $F$ , and let  $g(x) \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ . Set*

$$\phi_g(x, z) = g^{-1}(\phi(g(x), g(z))),$$

*an associate of  $F_g$  (recall Proposition 2.8). Let  $V$  be a nonlocal vertex  $F$ -algebra and let  $(W, Y_W)$  be a  $\phi$ -coordinated (resp. quasi)  $V$ -module. For  $v \in V$ , set*

$$Y_W^g(v, x) = Y_W(v, g(x)) \in (\text{End } W)[[x, x^{-1}]].$$

*Then  $(W, Y_W^g)$  carries the structure of a  $\phi_g$ -coordinated (resp. quasi)  $V_g$ -module, where  $V_g$  is a nonlocal vertex  $F_g$ -algebra with  $V_g = V$  as a vector space and with  $Y_g(v, x) = Y(v, g(x))$  for  $v \in V$  (recall Proposition 3.6).*

*Proof.* We shall consider the quasi case, while the non-quasi case will be clear from the proof. We have

$$\begin{aligned} Y_W^g(v, x) &= Y_W(v, g(x)) \in \text{Hom}(W, W((x))) \quad \text{for } v \in V, \\ Y_W^g(\mathbf{1}, x) &= Y_W(\mathbf{1}, g(x)) = 1_W. \end{aligned}$$

For  $u, v \in V$ , there exists  $q(z_1, z_2) \in \mathbb{C}[[z_1, z_2]]$  such that  $q(\phi(z_2, z_0), z_2) \neq 0$ ,

$$q(z_1, z_2)Y_W(u, z_1)Y_W(v, z_2) \in \text{Hom}(W, W((z_1, z_2))),$$

and

$$(q(z_1, z_2)Y_W(u, z_1)Y_W(v, z_2))|_{z_1=\phi(z_2, z_0)} = q(\phi(z_2, z_0), z_2)Y_W(Y(u, z_0)v, z_2).$$

With substitution  $z_i = g(x_i)$  for  $i = 0, 1, 2$ , we get

$$q(g(x_1), g(x_2))Y_W(u, g(x_1))Y_W(v, g(x_2)) \in \text{Hom}(W, W((x_1, x_2)))$$

and

$$\begin{aligned} &[q(g(x_1), g(x_2))Y_W(u, g(x_1))Y_W(v, g(x_2))]|_{g(x_1)=\phi(g(x_2), g(x_0))} \\ &= q(\phi(g(x_2), g(x_0)), g(x_2))Y_W(Y(u, g(x_0))v, g(x_2)). \end{aligned}$$

Notice that the substitution  $x_1 = \phi_g(x_2, x_0)$  amounts to  $g(x_1) = \phi(g(x_2), g(x_0))$ . Then

$$\begin{aligned} &[q(g(x_1), g(x_2))Y_W^g(u, x_1)Y_W^g(v, x_2)]|_{x_1=\phi_g(x_2, x_0)} \\ &= q(g(\phi_g(x_2, x_0)), g(x_2))Y_W^g(Y_g(u, x_0)v, x_2). \end{aligned}$$

We have  $q(g(x_1), g(x_2)) \in \mathbb{C}[[x_1, x_2]]$  such that

$$q(g(\phi_g(x, z)), g(x)) = q(\phi(g(x), g(z)), g(x)) \neq 0.$$

This proves that  $(W, Y_W^g)$  is a  $\phi_g$ -coordinated quasi  $V_g$ -module. □

Recall that for a nonlocal vertex algebra  $V$ ,  $V_F$  is a nonlocal vertex  $F$ -algebra with  $V_F = V$  as a vector space and  $Y_F(v, x) = Y(v, f(x))$  for  $v \in V$ . With Proposition 4.4 the following is immediate:

**Corollary 4.5.** *Let  $V$  be a nonlocal vertex algebra, let  $F$  be a formal group with  $f = \log F$ , and let  $\phi$  be an associate of  $F_a$ . Set*

$$\bar{\phi}(x, z) = f^{-1}(\phi(f(x), f(z))),$$

*an associate of  $F$ . The map, sending  $(W, Y_W)$  to  $(W, Y_W^f)$  and sending each morphism  $\theta$  to itself, is an isomorphism from the category of  $\phi$ -coordinated (resp. quasi)  $V$ -modules to the category of  $\bar{\phi}$ -coordinated (resp. quasi)  $V_F$ -modules.*

The following is another connection between  $\phi$ -coordinated modules for nonlocal vertex algebras and nonlocal vertex  $F$ -algebras:

**Theorem 4.6.** *Let  $F$  be a formal group over  $\mathbb{C}$ , let  $\phi$  be an associate of  $F$ , and let  $g \in x\mathbb{C}[[x]]$  with  $g'(0) = 1$ . Set*

$$\hat{\phi}_g(x, z) = \phi(x, g(z)),$$

*an associate of  $F_g$  by Proposition 2.10. Let  $V$  be a nonlocal vertex algebra and let  $W$  be a vector space equipped with a linear map  $Y_W(\cdot, x) : V \rightarrow (\text{End } W)[[x, x^{-1}]]$ . Then  $(W, Y_W)$  is a  $\phi$ -coordinated (resp. quasi)  $V$ -module if and only if  $(W, Y_W)$  is a  $\hat{\phi}_g$ -coordinated (resp. quasi) module for nonlocal vertex  $F_g$ -algebra  $(V, Y_g, \mathbf{1})$ .*

*Proof.* We shall just consider the quasi case, as the non-quasi case will be clear from the proof. Note that the  $\phi$ -coordinated quasi  $V$ -module structure and the  $\hat{\phi}_g$ -coordinated quasi  $V_g$ -module structure have the same requirement that  $Y_W(v, x) \in \text{Hom}(W, W((x)))$  for  $v \in V$  and  $Y_W(\mathbf{1}, x) = 1_W$ . Let  $u, v \in V$ . Assume

$$q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

for some  $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ . We see that  $q(\phi(x_2, x_0), x_2) \neq 0$  if and only if  $q(\phi(x_2, g(x_0)), x_2) \neq 0$ . Furthermore, a  $\phi$ -coordinated quasi  $V$ -module structure requires

$$(q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))_{x_1=\phi(x_2, x_0)} = q(\phi(x_2, x_0), x_2)Y_W(Y(u, x_0)v, x_2),$$

while a  $\phi_g$ -coordinated quasi  $V_g$ -module structure requires

$$(q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))_{x_1=\hat{\phi}_g(x_2, x_0)} = q(\hat{\phi}_g(x_2, x_0), x_2)Y_W(Y_g(u, x_0)v, x_2),$$

which amounts to

$$\begin{aligned} & (q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, g(x_0))} \\ &= q(\phi(x_2, g(x_0)), x_2)Y_W(Y(u, g(x_0))v, x_2). \end{aligned}$$

Then the equivalence between a  $\phi$ -coordinated quasi  $V$ -module structure and a  $\hat{\phi}_g$ -coordinated quasi  $V_g$ -module structure is immediate.  $\square$

Next, we give a construction of nonlocal vertex  $F$ -algebras and their  $\phi$ -coordinated (quasi) modules, by using the results of [Li9]. Let  $W$  be a vector space over  $\mathbb{C}$ . Set

$$\mathcal{E}(W) = \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]].$$

A finite sequence  $a^1(x), \dots, a^r(x)$  in  $\mathcal{E}(W)$  is said to be *quasi compatible* if there exists a nonzero power series  $p(x, y) \in \mathbb{C}[[x, y]]$  such that

$$\left( \prod_{1 \leq i < j \leq r} p(x_i, x_j) \right) a^1(x_1) \cdots a^r(x_r) \in \text{Hom}(W, W((x_1, \dots, x_r))). \quad (4.3)$$

It is said to be *compatible* if the above containment relation holds for  $p(x, y) = (x - y)^k$  for some  $k \in \mathbb{N}$ . We say a subset  $U$  of  $\mathcal{E}(W)$  is (*resp. quasi*) *compatible* if any finite sequence in  $U$  is (*resp. quasi*) compatible.

Let  $F$  be a formal group over  $\mathbb{C}$  and let  $\phi(x, z)$  be an associate of  $F$ . We define a notion of  $\phi$ -*quasi compatible* sequence (subset) by additionally requiring that  $p(\phi(x, z), x) \neq 0$  in the above definition. Notice that if  $\phi(x, z) \neq x$ , in view of Lemma 2.13,  $\psi$ -quasi compatibility is simply the same as quasi compatibility.

**Definition 4.7.** Let  $a(x), b(x) \in \mathcal{E}(W)$  be such that  $(a(x), b(x))$  is  $\phi$ -quasi compatible, i.e., there exists  $p(x, y) \in \mathbb{C}[[x, y]]$  with  $p(\phi(x, z), x) \neq 0$  such that

$$p(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (4.4)$$

We define  $a(x)_n^\phi b(x) \in \mathcal{E}(W)$  for  $n \in \mathbb{Z}$  in terms of the generating function

$$Y_{\mathcal{E}}^\phi(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n^\phi b(x) z^{-n-1}$$

by

$$Y_{\mathcal{E}}^\phi(a(x), z)b(x) = p(\phi(x, z), x)^{-1} \iota_{x, z} (p(x_1, x)a(x_1)b(x))|_{x_1=\phi(x, z)}, \quad (4.5)$$

where  $p(\phi(x, z), x)^{-1}$  denotes the inverse of  $p(\phi(x, z), x)$  in  $\mathbb{C}((x))((z))$ .

The same argument in [Li3] shows that  $Y_{\mathcal{E}}^\phi(a(x), z)b(x)$  is well defined; it is independent of the choice of  $p(x, y)$ .

A  $\psi$ -quasi compatible subspace  $U$  of  $\mathcal{E}(W)$  is said to be  $Y_{\mathcal{E}}^\phi$ -*closed* if

$$a(x)_n^\phi b(x) \in U \quad \text{for } a(x), b(x) \in U, n \in \mathbb{Z}.$$

The following generalizes the corresponding result of [Li9]:

**Theorem 4.8.** Let  $F$  be a formal group over  $\mathbb{C}$  and let  $\phi(x, z)$  be an associate of  $F$ . Let  $W$  be a vector space and let  $U$  be a  $\phi$ -quasi compatible subset of  $\mathcal{E}(W)$ . Then there exists a  $Y_{\mathcal{E}}^\phi$ -closed  $\phi$ -quasi compatible subspace which contains  $U$  and  $1_W$ . Denote by  $\langle U \rangle_\phi$  the smallest such  $Y_{\mathcal{E}}^\phi$ -closed  $\phi$ -quasi compatible subspace. Then  $(\langle U \rangle_\phi, Y_{\mathcal{E}}^\phi, 1_W)$  carries the structure of a nonlocal vertex  $F$ -algebra and  $W$  is a  $\phi$ -coordinated quasi  $\langle U \rangle_\phi$ -module with  $Y_W(a(x), z) = a(z)$  for  $a(x) \in \langle U \rangle_\phi$ .

*Proof.* Let  $f$  be the logarithm of  $F$ . Set

$$\tilde{\phi}(x, z) = \phi(x, f^{-1}(z)).$$

By Proposition 2.10,  $\tilde{\phi}(x, z)$  is an associate of  $F_a$ . For any  $q(x, y) \in \mathbb{C}[[x, y]]$ , we see that  $q(\phi(x, z), x) \neq 0$  if and only if  $q(\tilde{\phi}(x, z), x) = q(\phi(x, f^{-1}(z)), x) \neq 0$ . Then  $\phi$ -quasi compatibility is the same as  $\tilde{\phi}$ -quasi compatibility and hence  $U$  is  $\tilde{\phi}$ -quasi compatible. By Theorem 4.11 of [Li9], there exists a  $Y_{\mathcal{E}}^{\tilde{\phi}}$ -closed  $\tilde{\phi}$ -quasi compatible subspace containing  $U$  and  $1_W$ , and  $(\langle U \rangle_{\tilde{\phi}}, Y_{\mathcal{E}}^{\tilde{\phi}}, 1_W)$  carries the structure of a nonlocal vertex algebra with  $W$  as a  $\tilde{\phi}$ -coordinated quasi module with  $Y_W(a(x), z) = a(z)$  for  $a(x) \in \langle U \rangle_{\tilde{\phi}}$ , where  $\langle U \rangle_{\tilde{\phi}}$  is the smallest  $Y_{\mathcal{E}}^{\tilde{\phi}}$ -closed  $\tilde{\phi}$ -quasi compatible subspace of  $\mathcal{E}(W)$ , containing  $U$  and  $1_W$ .

Let  $(a(x), b(x))$  be a  $\phi$ -quasi compatible pair in  $\mathcal{E}(W)$ . There exists  $p(x, y) \in \mathbb{C}[[x, y]]$  with  $p(\phi(x, z), x) \neq 0$  such that

$$p(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

We have

$$\begin{aligned} Y_{\mathcal{E}}^{\tilde{\phi}}(a(x), z)b(x) &= p(\tilde{\phi}(x, z), x)^{-1} \iota_{x,z} (p(x_1, x)a(x_1)b(x))|_{x_1=\tilde{\phi}(x,z)}, \\ Y_{\mathcal{E}}^{\phi}(a(x), z)b(x) &= p(\phi(x, z), x)^{-1} \iota_{x,z} (p(x_1, x)a(x_1)b(x))|_{x_1=\phi(x,z)}. \end{aligned}$$

From this we get

$$Y_{\mathcal{E}}^{\phi}(a(x), z)b(x) = Y_{\mathcal{E}}^{\tilde{\phi}}(a(x), f(z))b(x). \quad (4.6)$$

By Theorem 3.7,  $(\langle U \rangle_{\tilde{\phi}}, Y_{\mathcal{E}}^{\tilde{\phi}}, 1_W)$  is a nonlocal vertex  $F$ -algebra. In particular,  $\langle U \rangle_{\tilde{\phi}}$  is  $Y_{\mathcal{E}}^{\tilde{\phi}}$ -closed. This proves the first assertion. It follows from (4.6) that  $\langle U \rangle_{\tilde{\phi}}$  is also the smallest  $Y_{\mathcal{E}}^{\phi}$ -closed  $\phi$ -quasi compatible subspace containing  $U$  and  $1_W$ . That is,  $\langle U \rangle_{\phi} = \langle U \rangle_{\tilde{\phi}}$ . Furthermore, by Theorem 4.6,  $(W, Y_W)$  is a  $\phi$ -coordinated quasi  $\langle U \rangle_{\phi}$ -module.  $\square$

## 5 $\phi$ -coordinated modules for nonlocal vertex algebras

In this section, we focus our attention on  $\phi$ -coordinated modules for nonlocal vertex algebras, in particular for ordinary vertex algebras, with a specialized  $\phi(x, z) = xe^z$ . For a nonlocal vertex  $\mathbb{Z}$ -graded algebra  $V$ , we exhibit a canonical connection between  $V$ -modules and  $\phi$ -coordinated modules for a nonlocal vertex algebra obtained from  $V$  by Zhu's change-of-variables theorem. Much of this section is motivated by the work of Lepowsky ([Le2], [Le3]).

Throughout this section, we fix

$$\phi(x, z) = xe^z,$$

a particular associate of the additive formal group  $F_a$ . We have (recall Lemma 2.13)

$$q(xe^z, x) \neq 0 \quad \text{in } \mathbb{C}((x))[[z]]$$

for every nonzero  $q(x, y) \in \mathbb{C}((x, y))$ .

Let  $V$  be a nonlocal vertex algebra. For a  $\phi$ -coordinated quasi  $V$ -module  $(W, Y_W)$ , the weak  $\phi$ -associativity states that for any  $u, v \in V$ , there exists  $0 \neq q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  such that

$$q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))) \quad (5.1)$$

and

$$q(x_2e^{x_0}, x_2)Y_W(Y(u, x_0)v, x_2) = (q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=x_2e^{x_0}}. \quad (5.2)$$

The following was obtained in [Li9] (Lemma 3.2):

**Lemma 5.1.** *Let  $V$  be a nonlocal vertex algebra and let  $(W, Y_W)$  be a  $\phi$ -coordinated quasi  $V$ -module. Then*

$$Y_W(e^{z\mathcal{D}}v, x) = Y_W(v, xe^z) = e^{zx\frac{d}{dx}}Y_W(v, x) \quad \text{for } v \in V, \quad (5.3)$$

where  $\mathcal{D}$  is the linear operator on  $V$ , defined by  $\mathcal{D}u = u_{-2}\mathbf{1}$  for  $u \in V$ . In particular,

$$Y_W(\mathcal{D}v, x) = x\frac{d}{dx}Y_W(v, x). \quad (5.4)$$

While nonlocal vertex algebras are too general, what we called weak quantum vertex algebras in [Li3] form a special class of nonlocal vertex algebras, which naturally generalize vertex algebras and vertex superalgebras. A *weak quantum vertex algebra* is a nonlocal vertex algebra  $V$  satisfying the condition that for  $u, v \in V$ , there exist

$$u^{(i)}, v^{(i)} \in V, f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

such that

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\sum_{i=1}^r \iota_{x_2, x_1}(f_i(x_2 - x_1))Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \\ & = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y(Y(u, x_0)v, x_2). \end{aligned} \quad (5.5)$$

We recall the following result from [Li9]:

**Proposition 5.2.** *Let  $V$  be a weak quantum vertex algebra and let  $(W, Y_W)$  be a  $\phi$ -coordinated module for  $V$  viewed as a nonlocal vertex algebra. Let  $u, v \in V$  and assume that*

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r \iota_{x_2, x_1}(f_i(e^{x_1 - x_2})) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1)$$

with  $k \in \mathbb{N}$ ,  $f_i(x) \in \mathbb{C}(x)$ ,  $u^{(i)}, v^{(i)} \in V$  for  $1 \leq i \leq r$ . Then

$$\begin{aligned} & (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) Y_W(u, x_1) Y_W(v, x_2) \\ & - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) \sum_{i=1}^r \iota_{x_2, x_1}(f_i(x_1/x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) \\ & = x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) Y_W(Y(u, \log(1+z))v, x_2). \end{aligned} \quad (5.6)$$

Furthermore, we have

$$\begin{aligned} & Y_W(u, x_1) Y_W(v, x_2) - \sum_{i=1}^r \iota_{x_2, x_1}(f_i(x_1/x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) \\ & = \text{Res}_{x_0} \delta \left( \frac{x_2 e^{x_0}}{x_1} \right) Y_W(Y(u, x_0)v, x_2). \end{aligned} \quad (5.7)$$

Note that

$$\delta \left( \frac{x_2 e^{x_0}}{x_1} \right) = e^{x_0 \left( x_2 \frac{\partial}{\partial x_2} \right)} \delta \left( \frac{x_2}{x_1} \right).$$

As an immediate consequence we have:

**Corollary 5.3.** *Let  $V$  be a vertex algebra and let  $(W, Y_W)$  be a  $\phi$ -coordinated module for  $V$  viewed as a nonlocal vertex algebra. Then*

$$[Y_W(u, x_1), Y_W(v, x_2)] = \sum_{j \geq 0} \frac{1}{j!} Y_W(u_j v, x_2) \left( x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left( \frac{x_2}{x_1} \right) \quad (5.8)$$

for  $u, v \in V$ .

Furthermore, using Corollary 5.3 and Lemma 5.1 we immediately get:

**Corollary 5.4.** *Let  $(V, Y, \mathbf{1}, \omega)$  be a vertex operator algebra of central charge  $\ell \in \mathbb{C}$  in the sense of [FLM] and let  $(W, Y_W)$  be a  $\phi$ -coordinated module for  $V$  viewed as a nonlocal vertex algebra. Then*

$$[L^\phi(m), L^\phi(n)] = (m - n) L^\phi(m + n) + \frac{\ell}{12} m^3 \delta_{m+n, 0} \quad (5.9)$$

for  $m, n \in \mathbb{Z}$ , where  $Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} L^\phi(n) x^{-n}$ .



**Definition 5.5.** Let  $\Gamma$  be a subgroup of the additive group  $\mathbb{R}$ , containing  $\mathbb{Z}$ . A *nonlocal vertex  $\Gamma$ -graded algebra* is a nonlocal vertex algebra  $V$  equipped with a  $\Gamma$ -grading  $V = \bigoplus_{n \in \Gamma} V_{(n)}$ , satisfying the conditions that  $\mathbf{1} \in V_{(0)}$  and that for  $v \in V_{(m)}$  with  $m \in \Gamma$ ,

$$v_n V_{(k)} \subset V_{(m+k-n-1)} \quad \text{for } n \in \mathbb{Z}, k \in \Gamma. \quad (5.10)$$

For a nonlocal vertex  $\Gamma$ -graded algebra  $V$ , define a linear operator  $L(0)$  on  $V$  by

$$L(0)|_{V_{(n)}} = n \quad \text{for } n \in \Gamma. \quad (5.11)$$

From [FHL], we have

$$\begin{aligned} L(0)\mathbf{1} &= 0, \\ x^{L(0)}Y(v, x_1)x^{-L(0)} &= Y(x^{L(0)}v, x_1), \\ e^{xL(0)}Y(v, x_1)e^{-xL(0)} &= Y(e^{xL(0)}v, e^x x_1) \quad \text{for } v \in V, \end{aligned}$$

where  $x^{L(0)}u = x^n u$  for  $u \in V_{(n)}$  with  $n \in \Gamma$ .

Furthermore, follow Zhu ([Z1], [Z2]) to define a linear map

$$Y[\cdot, x] : V \rightarrow (\text{End} V)[[x, x^{-1}]]$$

by

$$Y[v, x] = Y(e^{xL(0)}v, e^x - 1) \quad \text{for } v \in V. \quad (5.12)$$

The following slightly generalizes Zhu's change-of-variables theorem from vertex operator algebras to nonlocal vertex  $\Gamma$ -graded algebras (cf. [Z1], [Z2]):

**Proposition 5.6.** *Let  $V$  be a nonlocal vertex  $\Gamma$ -graded algebra. Then  $(V, Y[\cdot, x], \mathbf{1})$  carries the structure of a nonlocal vertex algebra. Furthermore, if  $V$  is a vertex algebra,  $(V, Y[\cdot, x], \mathbf{1})$  is also a vertex algebra.*

*Proof.* For  $v \in V$ , we have

$$\begin{aligned} Y[\mathbf{1}, x]v &= Y(e^{xL(0)}\mathbf{1}, e^x - 1)v = Y(\mathbf{1}, e^x - 1)v = v, \\ Y[v, x]\mathbf{1} &= Y(e^{xL(0)}v, e^x - 1)\mathbf{1} = e^{(e^x - 1)\mathcal{D}}e^{xL(0)}v \in V[[x]] \end{aligned}$$

and  $\lim_{x \rightarrow 0} Y[v, x]\mathbf{1} = v$ . Let  $u, v, w \in V$ . We have

$$\begin{aligned} Y[u, x_0 + x_2]Y[v, x_2]w &= Y(e^{(x_0+x_2)L(0)}u, e^{x_0+x_2} - 1)Y(e^{x_2L(0)}v, e^{x_2} - 1)w, \\ Y[Y[u, x_0]v, x_2]w &= Y(e^{x_2L(0)}Y(e^{x_0L(0)}u, e^{x_0} - 1)v, e^{x_2} - 1)w \\ &= Y(Y(e^{(x_2+x_0)L(0)}u, e^{x_2}(e^{x_0} - 1))e^{x_2L(0)}v, e^{x_2} - 1)w. \end{aligned}$$

Since  $e^{(x_0+x_2)L(0)}u \in V \otimes \mathbb{C}[[x_0, x_2]]$  and  $e^{x_2L(0)}v \in V \otimes \mathbb{C}[[x_2]]$ , there exists a nonnegative integer  $l$  such that

$$\begin{aligned} &(z_0 + z_2)^l Y(e^{(x_0+x_2)L(0)}u, z_0 + z_2)Y(e^{x_2L(0)}v, z_2)w \\ &= (z_0 + z_2)^l Y(Y(e^{(x_2+x_0)L(0)}u, z_0)e^{x_2L(0)}v, z_2)w. \end{aligned}$$

Then by substituting  $z_0 = e^{x_2}(e^{x_0} - 1)$ ,  $z_2 = e^{x_2} - 1$  we obtain

$$(e^{x_0+x_2} - 1)^l Y[u, x_0 + x_2] Y[v, x_2] w = (e^{x_0+x_2} - 1)^l Y[Y[u, x_0]v, x_2] w,$$

noticing that  $(e^{x_0+x_2} - 1) = e^{x_2}(e^{x_0} - 1) + (e^{x_2} - 1)$ . As  $e^{x_0+x_2} - 1 = (x_0 + x_2)g(x_0 + x_2)$  for some  $g(x) \in \mathbb{C}[[x]]$  with  $g(0) = 1$ , after cancellation we get

$$(x_0 + x_2)^l Y[u, x_0 + x_2] Y[v, x_2] w = (x_0 + x_2)^l Y[Y[u, x_0]v, x_2] w.$$

This proves that  $(V, Y[\cdot, x], \mathbf{1})$  carries the structure of a nonlocal vertex algebra.

Now, assume that  $V$  is a vertex algebra. We shall prove that  $(V, Y[\cdot, x], \mathbf{1})$  is also a vertex algebra by establishing weak commutativity. Let  $u, v \in V$ . As  $e^{xL(0)}u, e^{xL(0)}v \in V \otimes \mathbb{C}[[x]]$ , there exists  $k \in \mathbb{N}$  such that

$$\begin{aligned} & (y_1 - y_2)^k Y(e^{x_1 L(0)}u, y_1) Y(e^{x_2 L(0)}v, y_2) \\ &= (y_1 - y_2)^k Y(e^{x_2 L(0)}v, y_2) Y(e^{x_1 L(0)}u, y_1). \end{aligned}$$

Then

$$\begin{aligned} & (e^{x_1} - e^{x_2})^k Y(e^{x_1 L(0)}u, e^{x_1} - 1) Y(e^{x_2 L(0)}v, e^{x_2} - 1) \\ &= (e^{x_1} - e^{x_2})^k Y(e^{x_2 L(0)}v, e^{x_2} - 1) Y(e^{x_1 L(0)}u, e^{x_1} - 1). \end{aligned}$$

That is,

$$(e^{x_1} - e^{x_2})^k Y[u, x_1] Y[v, x_2] = (e^{x_1} - e^{x_2})^k Y[v, x_2] Y[u, x_1].$$

Writing  $e^{x_1} - e^{x_2} = (x_1 - x_2)f(x_1, x_2)$ , where  $f(x_1, x_2) = \sum_{n \geq 1} \frac{1}{n!} \frac{x_1^n - x_2^n}{x_1 - x_2}$ , we get

$$(x_1 - x_2)^k f(x_1, x_2)^k Y[u, x_1] Y[v, x_2] = (x_1 - x_2)^k f(x_1, x_2)^k Y[v, x_2] Y[u, x_1].$$

As  $f(x_1, x_2)^k$  is invertible in  $\mathbb{C}[[x_1, x_2]]$ , by cancellation we obtain

$$(x_1 - x_2)^k Y[u, x_1] Y[v, x_2] = (x_1 - x_2)^k Y[v, x_2] Y[u, x_1].$$

Therefore,  $(V, Y[\cdot, x], \mathbf{1})$  is a vertex algebra.  $\square$

**Remark 5.7.** Let  $(V, Y, \mathbf{1}, \omega)$  be a vertex operator algebra of central charge  $\ell \in \mathbb{C}$ . Set  $\tilde{\omega} = \omega - \frac{\ell}{24}\mathbf{1}$ . It was proved in [Z1] (cf. [Hu]) that  $(V, Y[\cdot, x], \mathbf{1}, \tilde{\omega})$  is a vertex operator algebra isomorphic to  $(V, Y, \mathbf{1}, \omega)$ .

The following is the main result of this section:

**Proposition 5.8.** *Let  $V$  be a nonlocal vertex  $\mathbb{Z}$ -graded algebra and let  $(W, Y_W)$  be a (resp. quasi) module for  $V$  viewed as a nonlocal vertex algebra. For  $v \in V$ , set*

$$X_W(v, x) = Y_W(x^{L(0)}v, x) \in \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]]. \quad (5.13)$$

*Then  $(W, X_W)$  carries the structure of a  $\phi$ -coordinated (resp. quasi) module with  $\phi(x, z) = xe^z$  for the nonlocal vertex algebra  $(V, Y[\cdot, x], \mathbf{1})$ .*

*Proof.* First, as  $V$  is  $\mathbb{Z}$ -graded, we have  $X_W(v, x)w = Y_W(x^{L(0)}v, x)w \in W((x))$  for  $v \in V, w \in W$ . Second, as  $L(0)\mathbf{1} = 0$  (with  $\mathbf{1} \in V_{(0)}$ ) we have

$$X_W(\mathbf{1}, x) = Y_W(x^{L(0)}\mathbf{1}, x) = Y_W(\mathbf{1}, x) = 1_W.$$

Furthermore, let  $u, v \in V$ . As  $x_1^{L(0)}u \in V \otimes \mathbb{C}[x_1, x_1^{-1}]$ ,  $x_2^{L(0)}v \in V \otimes \mathbb{C}[x_2, x_2^{-1}]$ , it is clear that there exists a nonzero power series  $p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  such that

$$p(x_1, x_2)X_W(u, x_1)X_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

On the other hand, there exists a nonzero power series  $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$  such that

$$q(x_1, x_2)Y_W((x_2e^{x_0})^{L(0)}u, x_1)Y_W(x_2^{L(0)}v, x_2) \in \text{Hom}(W, W((x_1, x_2)))[[x_0]],$$

and

$$\begin{aligned} & \left( q(x_1, x_2)Y_W((x_2e^{x_0})^{L(0)}u, x_1)Y_W(x_2^{L(0)}v, x_2) \right) \Big|_{x_1=x_2+z} \\ &= q(x_2+z, x_2)Y_W(Y((x_2e^{x_0})^{L(0)}u, z)x_2^{L(0)}v, x_2). \end{aligned} \quad (5.14)$$

We are going to apply substitution  $z = x_2(e^{x_0} - 1)$ . Note that for any  $m \in \mathbb{Z}$ ,

$$\begin{aligned} \iota_{x_2, x_0} [(x_2 + z)^m \Big|_{z=x_2(e^{x_0}-1)}] &= \sum_{i \geq 0} \binom{m}{i} x_2^{m-i} x_2^i (e^{x_0} - 1)^i \\ &= x_2^m \sum_{i \geq 0} \binom{m}{i} (e^{x_0} - 1)^i = x_2^m (e^{x_0})^m = x_1^m \Big|_{x_1=x_2e^{x_0}}. \end{aligned}$$

Then

$$\begin{aligned} & \left[ \left( q(x_1, x_2)Y_W((x_2e^{x_0})^{L(0)}u, x_1)Y_W(x_2^{L(0)}v, x_2) \right) \Big|_{x_1=x_2+z} \right] \Big|_{z=x_2(e^{x_0}-1)} \\ &= \left( q(x_1, x_2)Y_W(x_1^{L(0)}u, x_1)Y_W(x_2^{L(0)}v, x_2) \right) \Big|_{x_1=x_2e^{x_0}}, \end{aligned}$$

while

$$\begin{aligned} & \left[ q(x_2+z, x_2)Y_W \left( Y((x_2e^{x_0})^{L(0)}u, z)x_2^{L(0)}v, x_2 \right) \right] \Big|_{z=x_2(e^{x_0}-1)} \\ &= q(x_2e^{x_0}, x_2)Y_W \left( Y((x_2e^{x_0})^{L(0)}u, x_2(e^{x_0}-1))x_2^{L(0)}v, x_2 \right) \\ &= q(x_2e^{x_0}, x_2)Y_W \left( x_2^{L(0)}Y(e^{x_0L(0)}u, e^{x_0}-1)v, x_2 \right) \\ &= q(x_2e^{x_0}, x_2)X_W(Y[u, x_0]v, x_2). \end{aligned}$$

Consequently, by (5.14) we obtain

$$\begin{aligned} & (q(x_1, x_2)X_W(u, x_1)X_W(v, x_2)) \Big|_{x_1=x_2e^{x_0}} \\ &= q(x_2e^{x_0}, x_2)X_W(Y[u, x_0]v, x_2). \end{aligned}$$

This proves that  $(W, X_W)$  carries the structure of a  $\phi$ -coordinated quasi module with  $\phi(x, z) = xe^z$  for the nonlocal vertex algebra  $(V, Y[\cdot, x], \mathbf{1})$ . From the proof, it is also clear for the non-quasi case.  $\square$

**Remark 5.9.** Let  $V$  be a vertex  $\mathbb{Z}$ -graded algebra and let  $(W, Y_W)$  be a module for  $V$  viewed as a nonlocal vertex algebra. Combining Propositions 5.8 and 5.2 we have

$$[X_W(u, x_1), X_W(v, x_2)] = \text{Res}_{x_0} \delta \left( \frac{x_2 e^{x_0}}{x_1} \right) X_W(Y[u, x_0]v, x_2) \quad (5.15)$$

for  $u, v \in V$ . This recovers a result of Lepowsky ([Le3], Theorem 3.1). In fact, it was the formal (unrigorous) relation (2.18) in [Le3] that motivated the study in this section.

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